

# Remarks on cutoff phenomena for random walks on Hamming Schemes

Katsuhiko Kikuchi

## Abstract

The sequence of the simple random walks on Hamming schemes  $\{H(n, q)\}_{n=1}^{\infty}$  has a cutoff phenomenon for each integer  $q$  greater than or equal to 3. In this paper, for the sequence of simple random walks on Hamming schemes  $\{H(n, q)\}_{n=1}^{\infty}$  with  $q \geq 3$ , we give a simple majorant and a sharp minorant function for total variance distances between transition distributions and stationary distributions.

## 1 Introduction.

For many random walks on finite graphs, the transition distributions converge to the distributions of the equilibrium. Moreover, if we have useful majorant and minorant functions for the distance of them, we find the critical behavior of transition distributions, for example, the rapidity of decrease of the distances in the small range near the suitable time. Such the phenomenon and the time are called the cutoff phenomenon and the time to stationarity, respectively. In this paper, we give majorant and minorant functions for total variance distances between transition distributions and the distributions of the equilibrium for the simple random walks on Hamming schemes  $\{H(n, q)\}_{n=1}^{\infty}$  with  $q \geq 3$ .

Cutoff phenomenon is defined as follows. For a finite set  $X$ , we denote by  $M(X)$  the vector space of all complex-valued measures on  $X$ . Take two measures  $\mu, \nu \in M(X)$  on  $X$  and define the total variance distance  $\|\mu - \nu\|_{TV}$  by

$$\|\mu - \nu\|_{TV} = \max\{|\mu(S) - \nu(S)|; S \subset X\}.$$

Let  $(X, E_X)$  be a simple connected finite unordered graph without loops. For  $x, x' \in X$ , we say that  $x$  is adjacent to  $x'$  if the (unordered) pair  $\{x, x'\}$  belongs to  $E_X$  and write  $x \sim x'$ . The transition probability  $p(\cdot, \cdot)$  is a function on  $X \times X$  such that (i)  $p(x, x') \geq 0$ , (ii)  $p(x, x') > 0$  if and only if  $x \sim x'$ , and (iii)  $\sum_{x' \in X} p(x, x') = 1$  for any  $x \in X$ . For a nonnegative integer, we define transition probability  $p^{(k)}(\cdot, \cdot)$  after  $k$ -steps recursively, by

$$p^{(0)}(x, x') = \delta_{x, x'}, \quad p^{(k)}(x, x') = \sum_{y \in X} p^{(k-1)}(x, y)p(y, x'), \quad k \geq 1,$$

where  $\delta_{x,x'}$  is the Kronecker delta. Fix an element  $x^{(0)} \in X$  of  $X$  and put  $\nu^{*k}(\cdot) = p^{(k)}(x^{(0)}, \cdot)$ . Then, we see that  $\nu^{*k}$  is a probability measure on  $X$ . We say that the transition probability  $p(\cdot, \cdot)$  is ergodic if there exists an integer  $k_0$  such that  $p^{(k)}(x, x') > 0$  for any elements  $x, x' \in X$  and  $k \geq k_0$ . A probability measure  $\pi$  on  $X$  is stationary if  $\sum_{x' \in X} \pi(x')p(x', x) = \pi(x)$  for any  $x \in X$ . Let  $\{(X_n, E_{X_n})\}$  be a sequence of simple connected finite unordered graphs without loops,  $\{p_n(\cdot, \cdot)\}$  the transition probabilities on  $X_n$ ,  $\{x_n^{(0)}\}$  the fixted points and  $\{\pi_n\}$  the stationary probabilities. For sequences  $\{a_n\}, \{b_n\}$  of positive real numbers with  $\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = 0$ , the sequence of the Markov chains  $\{(X_n, E_{X_n})\}$  has an  $(a_n, b_n)$ -cutoff if there exist functions  $f_{\pm} : [0, +\infty) \rightarrow \mathbb{R}$  with  $\lim_{c \rightarrow +\infty} f_+(c) = 0$ ,  $\lim_{c \rightarrow +\infty} f_-(c) = 1$ , and for each  $c > 0$ , we have

$$\limsup_{n \rightarrow \infty} \|\nu_n^{*[a_n + cb_n]} - \pi_n\|_{TV} \leq f_+(c), \quad (1.1)$$

$$\liminf_{n \rightarrow \infty} \|\nu_n^{*[a_n - cb_n]} - \pi_n\|_{TV} \geq f_-(c), \quad (1.2)$$

where  $\lceil \alpha \rceil, \lfloor \alpha \rfloor$  denote the least integer greater than or equal to  $\alpha$ , the greatest integer less than or equal to  $\alpha$ , respectively (see [D1], [D2], [DS]).  $f_+$  and  $f_-$  are called the upper bound and the lower bound, and we often take as a monotone decreasing, a monotone increasing function, respectively. We remark that the existence of  $f_+$  does not imply the ergodicity of each random walk on the graph  $(X_n, E_{X_n})$  (see [H]). If we take a majorant function  $h_+$  for  $\|\nu_n^{*[a_n + cb_n]} - \pi_n\|_{TV}$  with  $n \geq n_0$  for some positive integer  $n_0$ , we have the ergodicity of the random walk on  $(X_n, E_{X_n})$  for any  $n$  with  $n \geq n_0$ .

Let  $n$  be a positive integer and  $q$  an integer with  $q \geq 2$ . we denote by  $[q]_0 = \{0, 1, \dots, q-1\}$  and  $[n] = \{1, 2, \dots, n\}$ . As a graph, the Hamming scheme  $H(n, q) = (X_n, E_{X_n})$  is a finite graph with the vertex set  $X_n = [q]_0^n$  and the edge set

$$E_{X_n} = \{\{x, x'\} \subset X_n; \#\{j \in [n]; x_j \neq x'_j\} = 1\},$$

where  $x = (x_1, \dots, x_n)$ ,  $x' = (x'_1, \dots, x'_n) \in X_n$ , and  $\#S$  is the cardinal number for a finite set  $S$ . The transition probability  $p_n(\cdot, \cdot)$  for  $(X_n, E_{X_n})$  is defined by

$$p_n(x, x') = \begin{cases} \frac{1}{n(q-1)}, & x \sim x', \\ 0, & \text{otherwise.} \end{cases}$$

Put  $x^{(0)} = (0, \dots, 0) \in H(n, q)$  and  $\nu_n(\cdot) = p_n(x^{(0)}, \cdot)$ . Then,  $\nu_n$  is a probability measure on  $X_n$ . The graph  $H(n, q)$  is ergodic if and only if  $q \geq 3$ . Hora gives in [H] the limit function  $f_{\pm}(c) = \text{Erf}\left(\frac{e^{\mp \frac{c}{2}}}{2\sqrt{2}}\right)$  of  $\|\nu_n^{*(a_n \pm cb_n)} - \pi_n\|_{TV}$ , where  $\text{Erf} : \mathbb{R} \rightarrow \mathbb{R}$  is the error function defined by  $\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ , and  $(a_n, b_n) = \left(\frac{n(q-1)}{2q} \log n(q-1), \frac{n(q-1)}{2q}\right)$ .

Candidates of the majorant functions for  $\|\nu_n^{*k} - \pi_n\|_{TV}$  are given by Diaconis and Hanlon in [DH], and by Diaconis and Ram in [DR]. In those papers, the simple random walk on a Hamming scheme  $H(n, q)$  are regarded as a special case of Metropolis chains on a hypercube  $H(n, 2)$ . Mizukawa gives in [M1] a majorant function for total variance distance with the different time to stationarity  $\frac{n(q-1)}{2q} \log q^n$  for the case  $q \geq 3$ , and in

[M2] a majorant and a minorant function for the sequences of random walks on Hamming schemes with staying.

For each integer  $q$  with  $q \geq 5$ , we give a simple majorant function of  $\|\nu_n^{*k} - \pi_n\|_{TV}$ .

**Theorem 1.1** *Assume that  $q \geq 5$ . Let  $k = \frac{n(q-1)}{2q}(\log n(q-1) + c)$  be an integer with  $c > 0$ . Then, we have*

$$\|\nu_n^{*k} - \pi_n\|_{TV}^2 \leq \frac{1}{4}(e^{e^{-c}} - 1). \quad (1.3)$$

The key of the proof of this theorem is that  $e^{-x} \geq |1 - x|$  for any real number  $x$  with  $x \leq \frac{5}{4}$ . We cannot adapt the proof for the case  $q = 3, 4$ . The obstruction is that  $|1 - x| \geq e^{-x}$  for a real number  $x$  with  $x \geq \frac{4}{3}$ . So we replace the majorant function.

**Theorem 1.2** (1) *Assume that  $q = 3$  and  $n \geq 3$ . For any integer  $k = \frac{n}{3}(\log 2n + c)$  with  $c > 0$ , we have*

$$\|\nu_n^{*k} - \pi_n\|_{TV}^2 \leq \frac{5}{2}(e^{e^{-c}} - 1). \quad (1.4)$$

(2) *Suppose that  $q = 4$  and  $n \geq 2$ . For each integer  $k = \frac{3n}{8}(\log 3n + c)$  with  $c > 0$ , one has*

$$\|\nu_n^{*k} - \pi_n\|_{TV}^2 \leq \frac{9}{4}(e^{e^{-c}} - 1). \quad (1.5)$$

While, we give minorant function for  $\|\nu_n^{*k} - \pi_n\|_{TV}$  as follows.

**Theorem 1.3** *Fix a positive real number  $c_0 > 0$ . For any positive real number  $b > 0$ , there exists a positive integer  $n_0$  such that  $c_0 \leq \log n_0(q-1)$ , and for any integer  $k = \frac{n(q-1)}{2q}(\log n(q-1) - c)$  with  $0 \leq c \leq c_0$ , we have*

$$\|\nu_n^{*k} - \pi_n\|_{TV} \geq 1 - (4q + b)e^{-c}. \quad (1.6)$$

The above theorem says that we can take a function  $1 - 4qe^{-c}$  as a lower function of  $\|\nu_n^{*k} - \pi_n\|_{TV}$ .

## 2 Hamming schemes.

In this section, we give notations of Hamming schemes, referring to [BI], [CST], [D1].

For a positive integer  $m$ , we denote by  $[m] = \{1, 2, \dots, m\}$ ,  $[m]_0 = \{0, 1, \dots, m-1\}$  and by  $\sharp S$  the cardinal number for a finite set  $S$ .

Let  $n$  be a positive integer and  $q$  an integer with  $q \geq 2$ . Put

$$H(n, q) = [q]_0^n = \{x = (x_1, \dots, x_n); x_j \in [q]_0 (1 \leq j \leq n)\}. \quad (2.1)$$

Take  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n) \in H(n, q)$  and define  $d(x, y)$  by

$$d(x, y) = \sharp\{j \in [n]; x_j \neq y_j\}. \quad (2.2)$$

For  $x, x' \in H(n, q)$ , we say that  $x'$  is *adjacent* to  $x$  if  $d(x, x') = 1$  and write  $x \sim x'$ . We fix an integer  $q$  with  $q \geq 2$  and denote by  $X_n = H(n, q)$  for simplicity. We call an unordered pair  $\{x, x'\} \subset X_n$  such that  $x \sim x'$  the *edge* of  $X_n$ . Put

$$E_{X_n} = \{\{x, x'\} \in X_n; x \sim x'\}. \quad (2.3)$$

Then the pair  $(X_n, E_{X_n})$  is a simple undirected finite graph without loops. We call  $H(n, q)$  the *Hamming scheme*.  $X_n$  and  $E_{X_n}$  are called the *vertex set* and the *edge set* of  $H(n, q)$ , respectively. We see that the distance  $d(\cdot, \cdot)$  on  $X_n$  coincides that derived from  $E_{X_n}$ .

We denote by  $S_m$  the symmetric group on  $[m]$  or  $[m]_0$  for a positive integer  $m$ . Let  $G_n = S_q \wr S_n = S_q^n \rtimes S_n$  denote the *wreath product* of  $S_q$  by  $S_n$  with the product

$$(\tau_1, \dots, \tau_n; \sigma)(\tau'_1, \dots, \tau'_n; \sigma') = (\tau_1 \tau'_{\sigma^{-1}(1)}, \dots, \tau_n \tau'_{\sigma^{-1}(n)}; \sigma \sigma'), \quad (2.4)$$

where  $\tau_j, \tau'_j \in S_q$  ( $1 \leq j \leq n$ ),  $\sigma, \sigma' \in S_n$ , and we regard  $S_q$  and  $S_n$  as symmetric groups acting on  $[q]_0$  and on  $[n]$ , respectively.  $G_n$  acts on  $X_n$  by

$$g \cdot x = (\tau_1(x_{\sigma^{-1}(1)}), \dots, \tau_n(x_{\sigma^{-1}(n)})), \quad (2.5)$$

where  $g = (\tau_1, \dots, \tau_n; \sigma) \in G_n$  and  $x = (x_1, \dots, x_n) \in X_n$ . The action of  $G_n$  on  $X_n$  is transitive. Put  $x^{(0)} = (0, \dots, 0) \in X_n$ . Then the stabilizer  $H_n$  of  $G_n$  at  $x^{(0)}$  is given by

$$H_n = S_{q-1} \wr S_n = \{(\tau_1, \dots, \tau_n; \sigma) \in G_n; \tau_j \in S_{q-1} \text{ for all } j \in [n]\}, \quad (2.6)$$

where we regard  $S_{q-1}$  as a symmetric group acting on  $[q-1]$ . For each integer  $j \in \{0, 1, \dots, n\}$  we put

$$x^{(j)} = (\overbrace{1, \dots, 1}^j, 0, \dots, 0) \in X_n. \quad (2.7)$$

Then,  $X_n = G_n/H_n$  and we have the  $H_n$ -orbit decomposition

$$X_n = \bigcup_{j=0}^n H_n \cdot x^{(j)}. \quad (2.8)$$

We see that  $x \sim x'$  implies that  $g \cdot x \sim g \cdot x'$  for any  $x, x' \in X_n$  and  $g \in G_n$ . Hence, for any  $j \in \{0, 1, \dots, n\}$ , we have that  $x \in H_n \cdot x^{(j)}$  if and only if  $d(x^{(0)}, x) = j$ . For  $j \in \{0, 1, \dots, n\}$ , put

$$g^{(j)} = (\overbrace{(0, 1), \dots, (0, 1)}^j, 1_{S_q}, \dots, 1_{S_q}; 1_{S_n}), \quad (2.9)$$

where  $(0, 1) \in S_q$  is the transposition of 0 and 1, and  $1_{S_q} \in S_q$ ,  $1_{S_n} \in S_n$  are the identity permutations. We see that  $g^{(j)} \cdot x^{(0)} = x^{(j)}$ . Hence, we have the decomposition

$G_n = \bigcup_{j=0}^n H_n g^{(j)} H_n$  of  $G_n$  into  $H_n$ -double cosets. Let  $L^1(G_n)$  denote the algebra of all functions on  $G_n$  with the convolution

$$f_1 * f_2(g) = \sum_{g' \in G_n} f_1(g(g')^{-1}) f_2(g') = \sum_{g' \in G_n} f_1(g') f_2((g')^{-1}g), \quad (2.10)$$

where  $f_1, f_2 \in L^1(G_n)$  and  $g \in G_n$ . We see that  $(G_n, H_n)$  is a *Gelfand pair*, that is, the subalgebra  $L^1(H_n \backslash G_n / H_n) \subset L^1(G_n)$  of all  $H_n$ -biinvariant functions on  $G_n$  is a commutative algebra since  $(g^{(j)})^{-1} = g^{(j)}$  for any  $j \in \{0, 1, \dots, n\}$  (see [CST], Example 4.3.2). We denote by  $L(X_n)$  the Hilbert space of all functions on  $X_n$  with the inner product

$$\langle f_1, f_2 \rangle_{L(X_n)} = \sum_{x \in X_n} f_1(x) \overline{f_2(x)}, \quad (2.11)$$

where  $f_1, f_2 \in L(X_n)$ , and write  $\|f\|_{L(X_n)} = \langle f, f \rangle_{L(X_n)}^{\frac{1}{2}}$  for  $f \in L(X)$ .  $G_n$  acts on  $L(X_n)$  by

$$(g \cdot f)(x) = f(g^{-1} \cdot x), \quad (2.12)$$

where  $g \in G_n$ ,  $f \in L(X_n)$  and  $x \in X_n$ . It is easy to show that the action is unitary.

Let  $W$  be a  $G_n$ -module. We denote by  $W_{H_n}$  the subspace of all  $H_n$ -invariant elements in  $W$ , that is,

$$W_{H_n} = \{w \in W; h \cdot w = w \text{ for all } h \in H_n\}. \quad (2.13)$$

The condition that  $(G_n, H_n)$  is a Gelfand pair indicates the properties of irreducible components appearing in  $L(X_n)$ .

**Lemma 2.1** *Let  $L(X_n) = \bigoplus_{\lambda \in \Lambda} V_\lambda$  be an irreducible decomposition of  $L(X_n)$ .*

- (1)  *$L(X_n)$  is multiplicity-free, that is,  $V_\lambda$  is not equivalent to  $V_{\lambda'}$  if  $\lambda \neq \lambda'$ .*
- (2) *For any  $\lambda \in \Lambda$ , we have  $\dim(V_\lambda)_{H_n} = 1$ .*

*Proof.* See [CST], Theorem 4.4.2, 4.6.2 for example. ■

Each irreducible component  $V_\lambda$  is called the *spherical representation* for  $(G_n, H_n)$ .

We construct irreducible components in  $L(X_n)$ . Take an integer  $a \in [q]_0$  and define a function  $\chi_a : [q]_0 \rightarrow \mathbb{C}$  by

$$\chi_a(x) = \zeta_q^{ax},$$

where  $x \in [q]_0$  and  $\zeta_q = \exp \frac{2\pi i}{q} \in \mathbb{C}$  is a primitive  $q$ -th root of 1 in  $\mathbb{C}$ . For  $x \in [q]_0$ , we have

$$\sum_{a=0}^{q-1} \chi_a(x) = \begin{cases} q, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

For  $a = (a_1, \dots, a_n) \in X_n$ , we define a function  $\chi_a : X_n \rightarrow \mathbb{C}$  by

$$\chi_a(x) = \prod_{j=1}^n \chi_{a_j}(x_j) = \zeta_q^{a_1 x_1 + \dots + a_n x_n}, \quad (2.14)$$

where  $x = (x_1, \dots, x_n) \in X_n$ . Then, we see that  $\{\chi_a \in L(X_n); a \in X_n\}$  is an orthogonal basis for  $L(X_n)$  and  $\|\chi_a\|_{L(X_n)} = q^{\frac{n}{2}}$  for  $a \in X_n$ .

For  $j \in \{0, 1, \dots, n\}$ , we put

$$V_j = \bigoplus_{\#\{l \in [n]; a_l \neq 0\} = j} \mathbb{C} \chi_a. \quad (2.15)$$

Then,  $V_j$  is  $G_n$ -invariant for each  $j$  with  $0 \leq j \leq n$  and we have the orthogonal desomposition

$$L(X_n) = \bigoplus_{j=0}^n V_j. \quad (2.16)$$

For  $j \in \{0, 1, \dots, n\}$ , we put

$$\omega_j = \sum_{\#\{l \in [n]; a_l \neq 0\} = j} \chi_a. \quad (2.17)$$

Then,  $\omega_j$  is nonzero  $H_n$ -invariant element in  $V_j$ , and any  $H_n$ -invariant element in  $V_j$  is the scalar multiple of  $\omega_j$ . Hence, all  $V_j$ 's are irreducible and  $L(X_n)$  is multiplicity-free.

For  $j \in \{0, 1, \dots, n\}$ , we define a function  $\phi_j : G_n \rightarrow \mathbb{C}$  by

$$\phi_j(g) = \left\langle \frac{\omega_j}{\|\omega_j\|_{L(X_n)}}, g \cdot \frac{\omega_j}{\|\omega_j\|_{L(X_n)}} \right\rangle_{L(X_n)} = \frac{1}{\|\omega_j\|_{L(X_n)}^2} \langle \omega_j, g \cdot \omega_j \rangle_{L(X_n)}. \quad (2.18)$$

Then,  $\phi_j$  is  $H_n$ -biinvariant and  $\phi_j(1_{G_n}) = 1$ , where  $1_{G_n} \in G_n$  is the unit element. Moreover,  $\phi_j$  is real-valued since  $(g^{(j)})^{-1} \in H_n g^{(j)} H_n$  for any  $j \in \{0, 1, \dots, n\}$  (see [CST], Theorem 4.8.2).  $\phi_j$  is called the *spherical function* on  $G_n$ . We regard  $\phi_j$  as an  $H_n$ -invariant function on  $X_n$ .  $\phi_j$  is calculated as

$$\phi_j(g^{(l)}) = \frac{1}{\binom{n}{j}} \sum_{r=0}^j \binom{l}{r} \binom{n-l}{j-r} \left(-\frac{1}{q-1}\right)^r, \quad (2.19)$$

where  $l \in \{0, 1, \dots, n\}$  (see [CST], Theorem 5.3.2). We give another realization of  $\phi_j(g^{(l)})$ . For a complex number  $\alpha$  and a nonnegative integer  $m$ , put

$$(\alpha)_m = \begin{cases} \alpha(\alpha+1) \cdots (\alpha+m-1), & m \geq 1, \\ 1, & m = 0. \end{cases}$$

$(\alpha)_m$  is called the *Pochhammer symbol*. Take complex numbers  $\alpha, \beta, \gamma \in \mathbb{C}$ , a variable  $x$ , and define the Gauss hypergeometric series

$$F \left( \begin{matrix} \alpha, \beta \\ \gamma \end{matrix} ; x \right) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m m!} x^m.$$

We write  $\phi_j(l) = \phi_j(g^{(l)})$  for simplicity. The polynomial  $\phi_j$  is called the *Krawtchouk polynomial*. Using a Gauss hypergeometric series,  $\phi_j(l)$  is realized as

$$\phi_j(l) = F \left( \begin{matrix} -j, -l \\ -n \end{matrix} ; \frac{q}{q-1} \right) = \sum_{r=0}^j \frac{(-j)_r (-l)_r}{(-n)_r r!} \left( \frac{q}{q-1} \right)^r, \quad (2.20)$$

where  $j, l \in \{0, 1, \dots, n\}$ .

For  $f \in L^1(H_n \backslash G_n / H_n)$ , we define

$$\widehat{f}(\phi_j) = \sum_{g \in G_n} f(g) \overline{\phi_j(g)} = \sum_{g \in G_n} f(g) \phi_j(g), \quad (2.21)$$

where  $j \in \{0, 1, \dots, n\}$ .  $\widehat{f}$  is called the *spherical transform* of  $f \in L^1(H_n \backslash G_n / H_n)$ . For  $f_1, f_2 \in L^1(H_n \backslash G_n / H_n)$ , We have

$$(f_1 * f_2)^\wedge = \widehat{f_1} \widehat{f_2}. \quad (2.22)$$

Take  $f \in L(X_n)_{H_n}$ ,  $j \in \{0, 1, \dots, n\}$  and define

$$\mathcal{F}(f)(\phi_j) = \sum_{x \in X_n} f(x) \overline{\phi_j(x)} = \sum_{x \in X_n} f(x) \phi_j(x). \quad (2.23)$$

$\mathcal{F}(f)$  is called the *spherical transform* of  $f \in L(X_n)_{H_n}$ .

For an  $H_n$ -invariant function  $f \in L(X_n)_{H_n}$  on  $X_n$ , we denote by  $\widetilde{f}$  the  $H_n$ -biinvariant function on  $G_n$  corresponding to  $f$ . We see that

$$\begin{aligned} \mathcal{F}(f)(\phi_j) &= \sum_{x \in X_n} f(x) \phi_j(x) = \sum_{x \in X_n} \frac{1}{\#H_n} \sum_{g \cdot x^{(0)} = x} f(g \cdot x^{(0)}) \phi_j(g) \\ &= \frac{1}{\#H_n} \sum_{g \in G_n} \widetilde{f}(g) \phi_j(g) = \frac{1}{\#H_n} (\widetilde{f})^\wedge(\phi_j). \end{aligned}$$

Take  $f_1, f_2 \in L(X_n)_{H_n}$  and define  $f_1 * f_2 \in L(X_n)_{H_n}$  such that

$$(f_1 * f_2)^\sim = \frac{1}{\#H_n} (\widetilde{f_1} * \widetilde{f_2}). \quad (2.24)$$

**Lemma 2.2** *Let  $f_1, f_2 \in L(X_n)_{H_n}$  be two elements in  $L(X_n)_{H_n}$ . For  $j \in \{0, 1, \dots, n\}$ , we have*

$$\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_1) \mathcal{F}(f_2). \quad (2.25)$$

*Proof.* By (2.22) and (2.23), we have

$$\begin{aligned} \mathcal{F}(f_1 * f_2)(\phi_j) &= \frac{1}{\#H_n} ((f_1 * f_2)^\sim)^\wedge(\phi_j) = \frac{1}{(\#H_n)^2} ((\widetilde{f_1}) * (\widetilde{f_2}))^\wedge(\phi_j) \\ &= \frac{1}{(\#H_n)^2} ((\widetilde{f_1})^\wedge(\widetilde{f_2})^\wedge)(\phi_j) = \mathcal{F}(f_1)(\phi_j) \mathcal{F}(f_2)(\phi_j). \end{aligned}$$

■

For  $f \in L(X_n)_{H_n}$  and a nonnegative integer  $k$ , we define  $f^{*k}$  recursively by

$$f^{*0} = \delta_{x^{(0)}}, \quad f^{*k} = f^{*(k-1)} * f, \quad k \geq 1. \quad (2.26)$$

**Lemma 2.3** *For  $j \in \{0, 1, \dots, n\}$  and a nonnegative integer  $k$ , we have*

$$\mathcal{F}(f^{*k})(\phi_j) = \mathcal{F}(f)(\phi_j)^k. \quad (2.27)$$

*Proof.* We prove it by induction in  $k$ . We see that

$$\mathcal{F}(f^{*0})(\phi_j) = \sum_{x \in X_n} \delta_{x^{(0)}}(x) \phi_j(x) = \phi_j(x^{(0)}) = 1.$$

We assume  $k \geq 1$  and the claim satisfies for any integer less than  $k$ . Then we have

$$\mathcal{F}(f^{*k})(\phi_j) = \mathcal{F}(f^{*(k-1)} * f) = \mathcal{F}(f^{*(k-1)})\mathcal{F}(f) = \mathcal{F}(f)^{k-1}\mathcal{F}(f) = \mathcal{F}(f)^k.$$

■

Take  $x, x' \in X_n$  and define the *transition probability*  $p_n(x, x')$  by

$$p_n(x, x') = \begin{cases} \frac{1}{n(q-1)}, & x \sim x', \\ 0, & x \not\sim x'. \end{cases} \quad (2.28)$$

Since  $G_n$  preserves adjacency on  $X_n$ , for  $x, x' \in X_n$  and  $g \in G_n$ , we see that

$$p_n(g \cdot x, g \cdot x') = p_n(x, x'). \quad (2.29)$$

Put

$$\nu_n(x) = p_n(x^{(0)}, x). \quad (2.30)$$

Then,  $\nu_n$  is an  $H_n$ -invariant probability measure on  $X_n$ .

**Lemma 2.4** *For  $j \in \{0, 1, \dots, n\}$ , we have*

$$\mathcal{F}(\nu_n)(\phi_j) = 1 - \frac{jq}{n(q-1)}. \quad (2.31)$$

*Proof.* Since  $x^{(0)} \sim x$  if and only if  $x \in H_n \cdot x^{(1)}$ , we see that

$$\phi_j(x^{(1)}) = F\left(\begin{matrix} -j, -1 \\ -n \end{matrix}; \frac{q}{q-1}\right) = 1 + \frac{(-j) \cdot (-1)}{(-n) \cdot 1} \cdot \frac{q}{q-1} = 1 - \frac{jq}{n(q-1)}.$$

Hence, we have

$$\begin{aligned} \mathcal{F}(\nu_n)(\phi_j) &= \sum_{x \in X_n} \nu_n(x) \phi_j(x) = \sum_{x^{(0)} \sim x} \nu_n(x) \phi_j(x) = \sum_{x \in H_n \cdot x^{(1)}} \nu_n(x) \phi_j(x^{(1)}) \\ &= n(q-1) \cdot \frac{1}{n(q-1)} \left(1 - \frac{jq}{n(q-1)}\right) = 1 - \frac{jq}{n(q-1)}. \end{aligned}$$

■

### 3 Upper bounds.

In this section, we give a majorant function for the distances between the  $k$ -step transitions distributions and the distributions of equilibrium for the simple random walks on the Hamming schemes  $\{H(n, q)\}$  with  $q \geq 3$ .

We denote by  $M(X_n)$  the vector space of all complex-valued measure on  $X_n$ . For a measure  $\mu \in M(X_n)$  on  $X_n$ , we put

$$\|\mu\|_{TV} = \max\{|\mu(S)|; S \subset X_n\}. \quad (3.1)$$



For two measure  $\mu, \nu \in M(X_n)$ , we define the *total variance distance* by  $\|\mu - \nu\|_{TV}$ . We regard a measure  $\mu \in M(X_n)$  on  $X_n$  as a function  $\mu : X_n \rightarrow \mathbb{C}$  defined by  $\mu(x) = \mu(\{x\})$  for  $x \in X_n$ . Take two probability measures  $\mu, \nu \in M(X_n)$  on  $X_n$ . Then we have an equality

$$\|\mu - \nu\|_{TV}^2 = \left( \frac{1}{2} \sum_{x \in X_n} |\mu(x) - \nu(x)| \right)^2 \leq \frac{\#X_n}{4} \|\mu - \nu\|_{L(X_n)}^2.$$

We denote by  $\pi_n$  the uniform probability measure on  $X_n$ , that is,

$$\pi_n(S) = \frac{\#S}{\#X_n} = \frac{\#S}{q^n}, \quad (3.2)$$

where  $S \subset X_n$  is a subset of  $X_n$ . We find the upper bound of the total variance distance  $\|\nu_n^{*k} - \mu_n\|_{TV}$  with the Fourier transforms of spherical functions.

**Lemma 3.1 (The upper bound lemma)** *For a nonnegative integer  $k$ , we have*

$$\|\nu_n^{*k} - \pi_n\|_{TV}^2 \leq \frac{1}{4} \sum_{j=1}^n d_j |\mathcal{F}(\nu_n)(\phi_j)|^2. \quad (3.3)$$

*Proof.* See [D1], Chapter 3B, Lemma 1 or [CST], Corollary 4.9.2. ■

Before estimating the total variance distance  $\|\nu_n^k - \pi_n\|_{TV}$ , we give two inequalities.

**Lemma 3.2** (1) *For a real number  $x$  such that  $x \leq \frac{5}{4}$ , we have  $e^{-x} \geq |1 - x|$ .*

(2) *If the real number  $x$  satisfies the condition  $x \geq \frac{4}{3}$ , one has  $e^{-x} \leq |1 - x|$ .*

*Proof.* First, we see that

$$\frac{8}{3} = \sum_{j=0}^3 \frac{1}{j!} \leq e = \sum_{j=0}^{\infty} \frac{1}{j!} \leq \frac{5}{2} + \sum_{j=3}^{\infty} \frac{1}{2 \cdot 3^{j-2}} = \frac{5}{2} + \frac{1}{4} = \frac{11}{4}.$$

(1) By Taylor's theorem, for  $x \in \mathbb{R}$ , there exists a real number  $\theta \in \mathbb{R}$  with  $0 < \theta < 1$  such that

$$e^{-x} = 1 - x + \frac{x^2}{2} e^{-\theta x}.$$

If  $x \leq 1$ , we have  $e^{-x} \geq 1 - x = |1 - x|$ . Hence, we only consider the case  $1 \leq x \leq \frac{5}{4}$ . We see that

$$e^5 \leq \left( \frac{11}{4} \right)^5 = \frac{161051}{1024} \leq 256 = 4^4.$$

So  $e^{-\frac{5}{4}} \geq \frac{1}{4} = \frac{5}{4} - 1$ . Therefore, for any real number  $x$  such that  $1 \leq x \leq \frac{5}{4}$ , we have

$$e^{-x} \geq e^{-\frac{5}{4}} \geq \frac{5}{4} - 1 \geq x - 1 = |1 - x|.$$

(2) We see that

$$e^4 \geq \left(\frac{8}{3}\right)^4 = \frac{4096}{81} \geq 27 = 3^3.$$

Hence,  $e^{-\frac{4}{3}} \leq \frac{1}{3} = \frac{4}{3} - 1$ . This implies that for any real number  $x$  with  $x \geq \frac{4}{3}$ ,

$$e^{-x} \leq e^{-\frac{4}{3}} \leq \frac{4}{3} - 1 \leq x - 1 = |1 - x|.$$

■

Here, we give a majorant function for the total variance distance  $\|\nu_n^k - \pi_n\|_{TV}$  with a large integer  $q$ .

**Theorem 3.3** *Let  $q$  be an integer with  $q \geq 5$ . Take a positive integer  $k$  such that  $k = \frac{n(q-1)}{2q}(\log n(q-1) + c)$  with  $c > 0$ . Then, we have*

$$\|\nu_n^{*k} - \pi_n\|_{TV}^2 \leq \frac{1}{4}(e^{e^{-c}} - 1).$$

*Proof.* The condition  $q \geq 5$  implies that  $\frac{jq}{n(q-1)} \leq \frac{q}{q-1} \leq \frac{5}{4}$  for any integer  $j \in \{0, 1, \dots, n\}$ . Hence, by Lemma 3.1, we have

$$\begin{aligned} \|\nu_n^{*k} - \pi_n\|_{TV}^2 &\leq \frac{1}{4} \sum_{j=1}^n d_j |\mathcal{F}(\nu_n)(\phi_j)|^{2k} = \frac{1}{4} \sum_{j=1}^n \binom{n}{j} (q-1)^j \left| 1 - \frac{jq}{n(q-1)} \right|^{2k} \\ &\leq \frac{1}{4} \sum_{j=1}^n \frac{n^j (q-1)^j}{j!} e^{-\frac{2jkq}{n(q-1)}} = \frac{1}{4} \sum_{j=1}^n \frac{1}{j!} e^{j(\log n(q-1) - \frac{2kq}{n(q-1)})}. \end{aligned}$$

Since  $k = \frac{n(q-1)}{2q}(\log n(q-1) + c)$ , we have

$$\|\nu_n^{*k} - \pi_n\|_{TV}^2 \leq \frac{1}{4} \sum_{j=1}^n \frac{e^{-cj}}{j!} \leq \frac{1}{4} \sum_{j=1}^{\infty} \frac{e^{-cj}}{j!} = \frac{1}{4}(e^{e^{-c}} - 1).$$

■

It remains to show the case  $q = 3, 4$ . For any real number  $\alpha \in \mathbb{R}$ , we denote by

$$\begin{aligned} \lfloor \alpha \rfloor &= \max\{m \in \mathbb{Z}; m \leq \alpha\}, \\ \lceil \alpha \rceil &= \min\{m \in \mathbb{Z}; m \geq \alpha\}. \end{aligned}$$

In order to estimate a total variance distances, where  $q = 3, 4$ , we give a lemma.

**Lemma 3.4** (1) *Let  $m$  be an integer with  $m \geq 2$  and  $l$  an integer such that  $0 \leq l \leq m$ . Put  $f_m(x) = \frac{x-m+2}{x+m} = 1 - \frac{2m-2}{x+m}$ . Then, we have*

$$\sum_{p=l}^{2m-l-1} f_m(p) \leq 2 \log \frac{3m-l}{l+m} \leq \log 9. \quad (3.4)$$

(2) Assume  $m$  is an integer with  $m \geq 2$  and  $l$  is an integer such that  $0 \leq l \leq \left\lfloor \frac{m}{2} \right\rfloor$ . Put

$f_m(x) = \frac{2x - m + 3}{x + m} = 2 - \frac{3m - 3}{x + m}$ . Then, one has

$$\sum_{p=l}^{m-l-1} f_m(p) \leq 3 \log \frac{2m-l}{l+m} \leq \log 8. \quad (3.5)$$

*Proof.* (1)  $f_m$  is a monotone increasing continuous function on the open interval  $(-m, +\infty)$ . Hence,

$$\begin{aligned} \sum_{p=l}^{2m-l-1} f_m(p) &\leq \int_l^{2m-l} f_m(x) dx = \int_l^{2m-l} \left( 1 - \frac{2m-2}{x+m} \right) dx \\ &= 2m - 2l - (2m-2) \log \frac{3m-l}{l+m}. \end{aligned}$$

Put

$$g_m(x) = x + m \log \frac{3m-x}{x+m}.$$

If  $0 \leq x \leq m$ , we have

$$g'_m(x) = 1 + m \left( \frac{-1}{3m-x} - \frac{1}{x+m} \right) = \frac{-(x-m)^2}{(3m-x)(x+m)} \leq 0.$$

Hence,  $g_m(l) \geq g_m(m) = m$  for  $l \in \{0, 1, \dots, m\}$ . Therefore,

$$\sum_{p=l}^{m-l-1} f_m(p) \leq 2m - 2g_m(l) + 2 \log \frac{3m-l}{l+m} \leq 2 \log \frac{3m-l}{l+m} \leq 2 \log 3 = \log 9.$$

(2) Similarly to (1), we have

$$\begin{aligned} \sum_{p=l}^{m-l-1} f_m(p) &\leq \int_l^{m-l} f_m(x) dx = \int_l^{m-l} \left( 2 - \frac{3m-3}{x+m} \right) dx \\ &= 2m - 4l - (3m-3) \log \frac{2m-l}{l+m}. \end{aligned}$$

Put

$$g_m(x) = 4x + 3m \log \frac{2m-x}{x+m}.$$

For a real number  $x$  with  $0 \leq x \leq \frac{m}{2}$ , we see that

$$g'_m(x) = 4 + 3m \left( \frac{-1}{2m-x} - \frac{1}{x+m} \right) = \frac{-(2x-m)^2}{(2m-x)(x+m)} \leq 0.$$

Hence,  $g_m(l) \geq g_m\left(\frac{m}{2}\right) = 2m$  for any integer  $l$  such that  $0 \leq l \leq \left\lfloor \frac{m}{2} \right\rfloor$ . Therefore,

$$\sum_{p=l}^{m-l-1} f_m(p) \leq 3 \log \frac{2m-l}{l+m} + 2m - g_m(l) \leq 3 \log \frac{2m-l}{l+m} \leq 3 \log 2 = \log 8.$$

■

**Lemma 3.5** (1) Put  $a_{n,j} = 2^j \binom{n}{j}$  for integers  $n$  and  $j$  such that  $n \geq 3$  and that  $0 \leq j \leq n$ . Then, for any integers  $m$  and  $l$  such that  $m \geq 2$  and that  $0 \leq l \leq m-1$ , we have

$$\frac{a_{3m-3,3m-l-3}}{a_{3m-3,l+m-1}} \leq \frac{a_{3m-2,3m-l-2}}{a_{3m-2,l+m-1}} \leq \frac{a_{3m-1,3m-l-1}}{a_{3m-1,l+m-1}} \leq 9. \quad (3.6)$$

(2) We put  $a_{n,j} = 3^j \binom{n}{j}$  for integers  $n$  and  $j$  such that  $n \geq 2$  and that  $0 \leq j \leq n$ . In this case, for any integer  $m$  and  $l$  such that  $m \geq 2$  and that  $0 \leq l \leq \left\lfloor \frac{m-1}{2} \right\rfloor$ , we see that

$$\frac{a_{2m-2,2m-l-2}}{a_{2m-2,l+m-1}} \leq \frac{a_{2m-1,2m-l-1}}{a_{2m-1,l+m-1}} \leq 8. \quad (3.7)$$

*Proof.* (1) Assume  $n = 3m-1$ , where  $m$  is an integer such that  $m \geq 2$ . For an integer  $l$  such that  $0 \leq l \leq m-1$ , we have

$$\begin{aligned} \frac{a_{3m-1,3m-l-1}}{a_{3m-1,l+m-1}} &= 2^{2m-2l} \frac{\binom{3m-1}{3m-l-1}}{\binom{3m-1}{l+m-1}} = 2^{2m-2l} \frac{(2m-l)(2m-l-1)\cdots(l+1)}{(3m-l-1)(3m-l-2)\cdots(l+m)} \\ &= \prod_{p=l}^{2m-l-1} \frac{2p+2}{p+m} = \prod_{p=l}^{2m-l-1} \left(1 + \frac{p-m+2}{p+m}\right) \\ &\leq \prod_{p=l}^{2m-l-1} \exp\left(\frac{p-m+2}{p+m}\right) = \exp \sum_{p=l}^{2m-l-1} \frac{p-m+2}{p+m}. \end{aligned}$$

By Lemma 3.4(1), we have

$$\frac{a_{3m-1,3m-l-1}}{a_{3m-1,l+m-1}} \leq \exp(\log 9) = 9.$$

Next, we consider the case  $n = 3m-2$ . We see that  $2(2m-l)-(3m-l-1) = m-l+1 > 0$  for an integer  $l$  with  $0 \leq l \leq m-1$ . Hence,

$$\begin{aligned} \frac{a_{3m-2,3m-l-2}}{a_{3m-2,l+m-1}} &= 2^{2m-2l-1} \frac{\binom{3m-2}{3m-l-2}}{\binom{3m-2}{l+m-1}} = \frac{3m-l-1}{2(2m-l)} \cdot \frac{a_{3m-1,3m-l-1}}{a_{3m-1,l+m-1}} \\ &\leq \frac{a_{3m-1,3m-l-1}}{a_{3m-1,l+m-1}}. \end{aligned}$$

Finally, we assume  $n = 3m-3$  for an integer  $m$  such that  $m \geq 2$ . For  $l \in \{0, 1, \dots, m-1\}$ , we see that  $2(2m-l-1)-(3m-l-2) = m-l > 0$  and that

$$\begin{aligned} \frac{a_{3m-3,3m-l-3}}{a_{3m-3,l+m-1}} &= 2^{2m-2l-2} \frac{\binom{3m-3}{3m-l-3}}{\binom{3m-3}{l+m-1}} = \frac{3m-l-2}{2(2m-l-1)} \frac{a_{3m-1,3m-l-2}}{a_{3m-2,l+m-1}} \\ &\leq \frac{a_{3m-2,3m-l-2}}{a_{3m-2,l+m-1}}. \end{aligned}$$

(2) Assume  $n = 2m - 1$  for some integer  $m$  with  $m \geq 2$ . For an integer  $j$  such that  $0 \leq l \leq \left\lfloor \frac{m-1}{2} \right\rfloor$ , we have

$$\begin{aligned} \frac{a_{2m-1,2m-l-1}}{a_{2m-1,l+m-1}} &= 3^{m-2l} \frac{\binom{2m-1}{2m-l-1}}{\binom{2m-1}{l+m-1}} = 3^{m-2l} \frac{(m-l)(m-l-1) \cdots (l+1)}{(2m-l-1)(2m-l-2) \cdots (l+m)} \\ &= \prod_{p=l}^{m-l-1} \frac{3p+3}{p+m} = \prod_{p=l}^{m-l-1} \left( 1 + \frac{2p-m+3}{p+m} \right) \\ &\leq \prod_{p=l}^{m-l-1} \exp \left( \frac{2p-m+3}{p+m} \right) = \exp \sum_{p=l}^{m-l-1} \frac{2p-m+3}{p+m}. \end{aligned}$$

By Lemma 3.4 (2), we have

$$\frac{a_{2m-1,2m-l-1}}{a_{2m-1,l+m-1}} \leq \exp(\log 8) = 8.$$

Next, we consider the case  $n = 2m - 2$  for some integer  $m$  with  $m \geq 2$ . For an integer  $l$  such that  $0 \leq l \leq \left\lfloor \frac{m-1}{2} \right\rfloor$ , we have

$$\begin{aligned} \frac{a_{2m-2,2m-l-2}}{a_{2m-2,l+m-1}} &= 3^{m-2l-1} \frac{\binom{2m-2}{2m-l-2}}{\binom{2m-2}{l+m-1}} = \frac{2m-l-1}{3(m-l)} \cdot \frac{a_{2m-1,2m-l-1}}{a_{2m-1,l+m-1}} \\ &\leq \frac{a_{2m-1,2m-l-1}}{a_{2m-1,l+m-1}} \leq 8, \end{aligned}$$

since  $3(m-l) - (2m-l-1) = m-2l+1 \geq 0$ . ■

Using these lemmas, we estimate total variance distance  $\|\nu_n^{*k} - \pi_n\|_{TV}$  for  $q = 3, 4$ .

**Theorem 3.6** Assume  $q = 3$ . For a positive integer  $k = \frac{n}{3}(\log 2n + c)$  with  $n \geq 3$  and  $c > 0$ , we have

$$\|\nu_n^{*k} - \pi_n\|_{TV}^2 \leq \frac{5}{2}(e^{e^{-c}} - 1). \quad (3.8)$$

*Proof.* By Lemma 3.1, we see that

$$\|\nu_n^{*k} - \pi_n\|_{TV}^2 \leq \frac{1}{4} \sum_{j=1}^n d_j |\mathcal{F}(\nu_n)(\phi_j)|^{2k} = \frac{1}{4} \sum_{j=1}^n 2^j \binom{n}{j} \left| 1 - \frac{3j}{2n} \right|^{2k}.$$

Put  $a_{n,j} = 2^j \binom{n}{j}$ . Assume  $n = 3m-1$ , where  $m$  is an integer such that  $m \geq 2$ . By Lemma 3.5 (1), for an integer  $l$  such that  $0 \leq l \leq m-1$ , we have  $\frac{a_{3m-1,3m-l-1}}{a_{3m-1,l+m-1}} \leq 9$ . Moreover, for

an integer  $l$  such that  $0 \leq l \leq m-1$ , we have  $3(3m-l-1) - 2(3m-1) = 3m-3l-1 > 0$ ,  $2(3m-1) - 3(l+m-1) = 3m-3l+1 > 0$  and

$$\begin{aligned} \left| 1 - \frac{3(3m-l-1)}{2(3m-1)} \right| &= \frac{3(3m-l-1)}{2(3m-1)} - 1 = \frac{3m-3l-1}{2(3m-1)} \\ &\leq \frac{3m-3l+1}{2(3m-1)} = 1 - \frac{3(l+m-1)}{2(3m-1)} = \left| 1 - \frac{3(l+m-1)}{2(3m-1)} \right|. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{j=1}^n 2^j \binom{n}{j} \left| 1 - \frac{3j}{2n} \right|^{2k} &\leq \sum_{j=1}^{m-2} a_{n,j} \left| 1 - \frac{3j}{2n} \right|^{2k} + a_{n,2m-1} \left| 1 - \frac{3(2m-1)}{2n} \right|^{2k} \\ &\quad + \sum_{l=0}^{m-1} (a_{3m-1,l+m-1} + a_{3m-1,3m-l-1}) \left| 1 - \frac{3(l+m-1)}{2(3m-1)} \right|^{2k} \\ &\leq 10 \sum_{j=1}^{2m-1} a_{n,j} \left| 1 - \frac{3j}{2n} \right|^{2k} \leq 10 \sum_{j=1}^{2m-1} \frac{2^j n^j}{j!} e^{-\frac{3jk}{n}} \\ &\leq 10 \sum_{j=1}^{\infty} \frac{1}{j!} e^{j(\log 2n - \frac{3k}{n})}. \end{aligned}$$

Next, we consider the case  $n = 3m-2$ . We see that  $\frac{a_{3m-2,3m-l-2}}{a_{3m-2,l+m-1}} \leq 9$  for an integer  $l$  such that  $0 \leq l \leq m-1$  by Lemma 3.5 (1). If  $0 \leq l \leq m-1$ , we have that  $3(3m-l-2) - 2(3m-2) = 3m-3l-2 > 0$ ,  $2(3m-2) - 3(l+m-1) = 3m-3l-1 > 0$  and that

$$\begin{aligned} \left| 1 - \frac{3(3m-l-2)}{2(3m-2)} \right| &= \frac{3(3m-l-2)}{2(3m-2)} - 1 = \frac{3m-3l-2}{2(3m-2)} \\ &\leq \frac{3m-3l-1}{2(3m-2)} = 1 - \frac{3(l+m-1)}{2(3m-2)} = \left| 1 - \frac{3(l+m-1)}{2(3m-2)} \right|. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{j=1}^n 2^j \binom{n}{j} \left| 1 - \frac{3j}{2n} \right|^{2k} &\leq \sum_{j=1}^{m-2} a_{n,j} \left| 1 - \frac{3j}{2n} \right|^{2k} \\ &\quad + \sum_{l=0}^{m-1} (a_{3m-2,l+m-1} + a_{3m-2,3m-l-2}) \left| 1 - \frac{3(l+m-1)}{2(3m-2)} \right|^{2k} \\ &\leq 10 \sum_{j=1}^{2m-2} a_{n,j} \left| 1 - \frac{3j}{2n} \right|^{2k} \leq 10 \sum_{j=1}^{2m-2} \frac{2^j n^j}{j!} e^{-\frac{3jk}{n}} \\ &\leq 10 \sum_{j=1}^{\infty} \frac{1}{j!} e^{j(\log 2n - \frac{3k}{n})} \end{aligned}$$

Finally, we assume  $n = 3m-3$  for an integer  $m$  such that  $m \geq 2$ . For  $l \in \{0, 1, \dots, m-1\}$ , we see that  $\frac{a_{3m-3,3m-l-3}}{a_{3m-3,l+m-1}} \leq 9$  by Lemma 3.5 (1). Moreover, for any integer  $j$  with  $0 \leq l \leq$

$m-2$ , we have  $(3m-l-3)-2(m-1) = m-l-1 > 0$ ,  $2(m-1)-(l+m+1) = m-l-1 > 0$  and

$$\begin{aligned} \left| 1 - \frac{3(3m-l-3)}{2(3m-3)} \right| &= \frac{3m-l-3}{2(m-1)} - 1 = \frac{m-l-1}{2(m-1)} \\ &= 1 - \frac{l+m-1}{2(m-1)} = \left| 1 - \frac{3(l+m-1)}{2(3m-3)} \right|. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{j=1}^n 2^j \binom{n}{j} \left| 1 - \frac{3j}{2n} \right|^{2k} &= \sum_{j=1}^{m-2} a_{n,j} \left| 1 - \frac{3j}{2n} \right|^{2k} + a_{n,2m-2} \left| 1 - \frac{3(2m-2)}{2(3m-3)} \right|^{2k} \\ &\quad + \sum_{l=0}^{m-2} (a_{3m-3,l+m-1} + a_{3m-3,3m-l-3}) \left| 1 - \frac{3(l+m-1)}{2(3m-3)} \right|^{2k} \\ &\leq 10 \sum_{j=1}^{2m-3} a_{n,j} \left| 1 - \frac{3j}{2n} \right|^{2k} \leq 10 \sum_{j=1}^{2m-3} \frac{2^j n^j}{j!} e^{-\frac{3jk}{n}} \\ &\leq 10 \sum_{j=1}^{\infty} \frac{1}{j!} e^{j(\log 2n - \frac{3k}{n})}. \end{aligned}$$

Therefore, for any integer  $n$  such that  $n \geq 3$ , we have

$$\|\nu_n^{*k} - \pi_n\|_{TV}^2 \leq \frac{1}{4} \cdot 10 \cdot \sum_{j=0}^{\infty} \frac{1}{j!} e^{j(\log 2n - \frac{3k}{n})} = \frac{5}{2} \sum_{j=1}^{\infty} \frac{e^{-cj}}{j!} = \frac{5}{2} (e^{e^{-c}} - 1).$$

■

**Theorem 3.7** Suppose  $q = 4$ . For a positive integer  $k = \frac{3n}{8}(\log 3n + c)$  with  $n \geq 2$  and  $c > 0$ , we have

$$\|\nu_n^{*k} - \pi_n\|_{TV}^2 \leq \frac{9}{4} (e^{e^{-c}} - 1). \quad (3.9)$$

*Proof.* Similarly to Theorem 3.6, we have

$$\|\nu_n^{*k} - \pi_n\|_{TV}^2 \leq \frac{1}{4} \sum_{j=1}^n d_j |\mathcal{F}(\nu_n)(\phi_j)|^{2k} = \frac{1}{4} \sum_{j=1}^n 3^j \binom{n}{j} \left| 1 - \frac{4j}{3n} \right|^{2k}.$$

Put  $a_{n,j} = 3^j \binom{n}{j}$ . Assume  $n = 2m - 1$  for some integer  $m$  with  $m \geq 2$ . By Lemma 3.5 (2), for an integer  $j$  such that  $0 \leq l \leq \left\lfloor \frac{m-1}{2} \right\rfloor$ , we have  $\frac{a_{2m-1,2m-l-1}}{a_{2m-1,l+m-1}} \leq 8$ . Moreover, if  $0 \leq l \leq \left\lfloor \frac{m-1}{2} \right\rfloor$ , we see that

$$\begin{aligned} 4(2m-l-1) &\geq 4 \left( 2m - \left( \frac{m-1}{2} \right) - 1 \right) = 6m - 2 \geq 3(2m-1) \\ &\geq 6m - 6 = 4 \left( \frac{m-1}{2} + m - 1 \right) \geq 4(l+m-1). \end{aligned}$$

So we have

$$\begin{aligned} \left| 1 - \frac{4(2m-l-1)}{3(2m-1)} \right| &= \frac{4(2m-l-1)}{3(2m-1)} - 1 = \frac{2m-4l-1}{3(2m-1)} \\ &\leq \frac{2m-4l+1}{3(2m-1)} = 1 - \frac{4(l+m-1)}{3(2m-1)} = \left| 1 - \frac{4(l+m-1)}{3(2m-1)} \right|. \end{aligned}$$

Hence, if  $m = 2r$  for some positive integer  $r$ , we have

$$\begin{aligned} \sum_{j=1}^n 3^j \binom{n}{j} \left| 1 - \frac{4j}{3n} \right|^{2k} &\leq \sum_{j=1}^{m-2} a_{n,j} \left| 1 - \frac{4j}{3n} \right|^{2k} + a_{n,3r-1} \left| 1 - \frac{4(3r-1)}{3n} \right|^{2k} \\ &\quad + \sum_{l=0}^{r-1} (a_{2m-1,l+m-1} + a_{2m-1,2m-l-1}) \left| 1 - \frac{4(l+m-1)}{3(2m-1)} \right|^{2k} \\ &\leq 9 \sum_{j=1}^{3r-1} a_{n,j} \left| 1 - \frac{4j}{3n} \right|^{2k} \leq 9 \sum_{j=1}^{3r-1} \frac{3^j n^j}{j!} e^{-\frac{8jk}{3n}} \leq 9 \sum_{j=1}^{\infty} \frac{1}{j!} e^{j(\log 3n - \frac{8k}{3n})}. \end{aligned}$$

If  $m = 2r + 1$  for a positive integer  $r$ , we have  $r = \frac{m-1}{2}$  and

$$\begin{aligned} \sum_{j=1}^n 3^j \binom{n}{j} \left| 1 - \frac{4j}{3n} \right|^{2k} &\leq \sum_{j=1}^{m-2} a_{n,j} \left| 1 - \frac{4j}{3n} \right|^{2k} \\ &\quad + \sum_{l=0}^r (a_{2m-1,l+m-1} + a_{2m-1,2m-l-1}) \left| 1 - \frac{4(l+m-1)}{3(2m-1)} \right|^{2k} \\ &\leq 9 \sum_{j=1}^{3r} a_{n,j} \left| 1 - \frac{4j}{3n} \right|^{2k} \leq 9 \sum_{j=1}^{3r} \frac{3^j n^j}{j!} e^{-\frac{8jk}{3n}} \leq 9 \sum_{j=1}^{\infty} \frac{1}{j!} e^{j(\log 3n - \frac{8k}{3n})}. \end{aligned}$$

Next, we consider the case  $n = 2m - 2$  for some integer  $m$  with  $m \geq 2$ . For an integer  $l$  such that  $0 \leq l \leq \left\lfloor \frac{m-2}{2} \right\rfloor$ , we have  $\frac{a_{2m-2,2m-l-2}}{a_{2m-2,l+m-1}} \leq 8$  by Lemma 3.5 (2), and that

$$\begin{aligned} 2(2m-l-2) &\geq 2 \left( 2m - \left( \frac{m-2}{2} \right) - 2 \right) = 3m-2 \geq 3(m-1) \\ &\geq 3m-4 = 2 \left( \left( \frac{m-2}{2} \right) + m-1 \right) \geq 2(l+m-1). \end{aligned}$$

Hence,

$$\begin{aligned} \left| 1 - \frac{4(2m-l-2)}{3(2m-2)} \right| &= \frac{2(2m-l-2)}{3m-3} - 1 = \frac{m-2l-1}{3m-3} \\ &= 1 - \frac{2(l+m-1)}{3m-3} = \left| 1 - \frac{4(l+m-1)}{3(2m-2)} \right|. \end{aligned}$$



If  $m = 2r$  for some positive integer  $r$ , we have  $n = 4r - 2$  and

$$\begin{aligned} \sum_{j=1}^n 3^j \binom{n}{j} \left| 1 - \frac{4j}{3n} \right|^{2k} &= \sum_{j=1}^{m-2} a_{n,j} \left| 1 - \frac{4j}{3n} \right|^{2k} \\ &\quad + \sum_{l=0}^{r-1} (a_{2m-2,l+m-1} + a_{2m-2,2m-l-2}) \left| 1 - \frac{4(l+m-1)}{3(2m-2)} \right|^{2k} \\ &\leq 9 \sum_{j=1}^{3r-2} a_{n,j} \left| 1 - \frac{4j}{3n} \right|^{2k} \leq 9 \sum_{j=1}^{3r-2} \frac{3^j n^j}{j!} e^{-\frac{8jk}{3n}} \leq 9 \sum_{j=1}^{\infty} \frac{1}{j!} e^{j(\log 3n - \frac{8k}{3n})}. \end{aligned}$$

If there exists a positive integer  $r$  such that  $m = 2r + 1$ , then  $n = 4r$ ,  $r = \frac{m-1}{2}$  and

$$\begin{aligned} \sum_{j=1}^n 3^j \binom{n}{j} \left| 1 - \frac{4j}{3n} \right|^{2k} &= \sum_{j=1}^{m-2} a_{n,j} \left| 1 - \frac{4j}{3n} \right|^{2k} + a_{2m-2,3r} \left| 1 - \frac{4 \cdot 3r}{3 \cdot 4r} \right|^{2k} \\ &\quad + \sum_{l=0}^{r-1} (a_{2m-2,l+m-1} + a_{2m-2,2m-l-2}) \left| 1 - \frac{4(l+m-1)}{3(2m-2)} \right|^{2k} \\ &\leq 9 \sum_{j=1}^{3r-1} a_{n,j} \left| 1 - \frac{4j}{3n} \right|^{2k} \leq 9 \sum_{j=1}^{3r-1} \frac{3^j n^j}{j!} e^{-\frac{8jk}{3n}} \leq 9 \sum_{j=1}^{\infty} \frac{1}{j!} e^{j(\log 3n - \frac{8k}{3n})}. \end{aligned}$$

Therefore, for any integer  $n$  with  $n \geq 2$ , we have

$$\|\nu_n^{*k} - \pi_n\|_{TV}^2 \leq \frac{1}{4} \cdot 9 \sum_{j=1}^{\infty} \frac{1}{j!} e^{j(\log 3n - \frac{8k}{3n})} = \frac{9}{4} \sum_{j=1}^{\infty} \frac{e^{-cj}}{j!} = \frac{9}{4} (e^{e^{-c}} - 1).$$

■

**Remark 3.8** Diaconis and Ram study the analysis on the deformation of Markov chains on Coxeter groups, called the *Metropolis algorithm*. The random walks on the Hamming schemes  $\{H(n, q)\}_{n=1}^{\infty}$  are the special cases of them on the hypercubes  $\{H(n, 2)\}_{n=1}^{\infty}$  with  $\theta = \frac{1}{q-1}$  ([DR], Theorem 5.4). However, our majorant functions for the total variance distances are simpler than that they give.

## 4 Lower bounds.

In this section we give a minorant function for the total variance distance  $\|\nu_n^{*k} - \pi_n\|_{TV}$ . Let  $\mu$  be a probability measure on  $X_n$  and  $f$  a function on  $X_n$ . We denote by  $E_{\mu}(f)$  the expectation of  $f$  with respect to  $\mu$ , that is,

$$E_{\mu}(f) = \sum_{x \in X_n} f(x) \mu(x).$$

The variance  $Var_{\mu}(f)$  of  $f$  respect to  $\mu$  is defined by

$$Var_{\mu}(f) = E_{\mu}((f - E_{\mu}(f))^2) = E_{\mu}(f^2) - E_{\mu}(f)^2.$$

In order to compute expectations and variances of spherical functions, we need the precise description of  $\phi_j$ 's. We see that  $\phi_0 = 1$ , the constant function with the value 1. The spherical function  $\phi_1$  is described as

$$\phi_1(l) = 1 + \frac{(-1) \cdot (-l)}{(-n) \cdot 1} \cdot \frac{q}{q-1} = 1 - \frac{lq}{n(q-1)},$$

where  $l \in \{0, 1, \dots, n\}$ . We calculate the spherical function  $\phi_2$  to compute the variance of  $\phi_1$ . For  $l \in \{0, 1, \dots, n\}$ , we have

$$\begin{aligned} \phi_2(l) &= 1 + \frac{(-2) \cdot (-l)}{(-n) \cdot 1} \cdot \frac{q}{q-1} + \frac{(-2)_2(-l)_2}{(-n)_2 \cdot 2!} \cdot \left(\frac{q}{q-1}\right)^2 \\ &= 1 - \frac{2lq}{n(q-1)} + \frac{l(l-1)q^2}{n(n-1)(q-1)^2} \\ &= 1 - \frac{lq((2n-1)q - (2n-2))}{n(n-1)(q-1)^2} + \frac{l^2q^2}{n(n-1)(q-1)^2}. \end{aligned}$$

**Lemma 4.1** *We have*

$$\phi_1^2 = \frac{1}{n(q-1)}\phi_0 + \frac{q-2}{n(q-1)}\phi_1 + \frac{n-1}{n}\phi_2. \quad (4.1)$$

*Proof.* For  $l \in \{1, \dots, n\}$ , we see that

$$\phi_1(l)^2 = \left(1 - \frac{lq}{n(q-1)}\right)^2 = 1 - \frac{2lq}{n(q-1)} + \frac{l^2q^2}{n^2(q-1)^2}.$$

On the other hand, we have

$$\begin{aligned} &\frac{1}{n(q-1)}\phi_0(l) + \frac{q-2}{n(q-1)}\phi_1(l) + \frac{n-1}{n}\phi_2(l) \\ &= \frac{1}{n(q-1)} + \frac{q-2}{n(q-1)} \left(1 - \frac{lq}{n(q-1)}\right) \\ &\quad + \frac{n-1}{n} \left(1 - \frac{lq((2n-1)q - (2n-2))}{n(n-1)(q-1)^2} + \frac{l^2q^2}{n(n-1)(q-1)^2}\right) \\ &= \frac{1 + (q-2) + (n-1)(q-1)}{n(q-1)} - \frac{lq((q-2) + ((2n-1)q - (2n-2)))}{n^2(q-1)^2} + \frac{l^2q^2}{n^2(q-1)^2} \\ &= 1 - \frac{2lq}{n(q-1)} + \frac{l^2q^2}{n^2(q-1)^2}, \end{aligned}$$

it completes the proof of the lemma. ■

Here, we compute the expectations of  $\phi_j$ 's and the variance of  $\phi_1$  with respect to  $\pi_n$ .

**Lemma 4.2** (1) *We have*

$$E_{\pi_n}(\phi_j) = \begin{cases} 1, & j = 1, \\ 0, & 1 \leq j \leq n. \end{cases} \quad (4.2)$$

(2) *One has  $\text{Var}_{\pi_n}(\phi_1) = E_{\pi_n}(\phi_1^2) = \frac{1}{n(q-1)}$ .*

*Proof.* (1) We see that

$$E_{\pi_n}(\phi_0) = \sum_{x \in X_n} \phi_0(x) \pi_n(x) = (\#X_n) \cdot 1 \cdot \frac{1}{\#X_n} = 1.$$

We compute  $E_{\pi_n}(\phi_j)$  for  $j \in \{1, 2, \dots, n\}$ . For each  $x \in X_n$ , there exists  $l \in \{0, 1, \dots, n\}$  such that  $x \in H_n \cdot x^{(l)}$ . We see that

$$\#(H_n \cdot x^{(l)}) = \#\{x \in X_n; d(x^{(0)}, x) = l\} = (q-1)^l \binom{n}{l}.$$

Since  $\#X_n = q^n$ , we have

$$\begin{aligned} (\#X_n)E_{\pi_n}(\phi_j) &= (\#X_n) \sum_{x \in X_n} \phi_j(x) \pi_n(x) = (\#X_n) \cdot \frac{1}{\#X_n} \sum_{l=0}^n \sum_{x \in H_n \cdot x^{(l)}} \phi_j(x) \\ &= \sum_{l=0}^n (q-1)^l \binom{n}{l} \sum_{r=0}^j \frac{(-j)_r (-l)_r}{(-n)_r r!} \left(\frac{q}{q-1}\right)^r \\ &= \sum_{r=0}^j \frac{(-j)_r}{(-n)_r r!} \left(\frac{q}{q-1}\right)^r \sum_{l=0}^n (q-1)^l (-l)_r \binom{n}{l}. \end{aligned}$$

For  $r \in \{0, 1, \dots, j\}$ , we see that

$$\begin{aligned} \sum_{l=0}^n (q-1)^l (-l)_r \binom{n}{l} &= (-1)^r n(n-1) \cdots (n-r+1) (q-1)^r \sum_{l=r}^n (q-1)^{l-r} \binom{n-r}{l-r} \\ &= (-n)_r (q-1)^r \sum_{l=0}^{n-r} (q-1)^l \binom{n-r}{l} = (-n)_r q^{n-r} (q-1)^r. \end{aligned}$$

Hence, we have

$$\begin{aligned} (\#X_n)E_{\pi_n}(\phi_j) &= \sum_{r=0}^j \frac{(-j)_r}{(-n)_r r!} \left(\frac{q}{q-1}\right)^r \cdot (-n)_r q^{n-r} (q-1)^r \\ &= q^n \sum_{r=0}^j \frac{(-j)_r}{r!} = q^n \sum_{r=0}^j (-1)^r \binom{j}{r} = 0. \end{aligned}$$

(2) By Lemma 4.1, we have

$$\begin{aligned} Var_{\pi_n}(\phi_1) &= E_{\pi_n}(\phi_1^2) - E_{\pi_n}(\phi_1)^2 = E_{\pi_n}(\phi_1^2) \\ &= \frac{1}{n(q-1)} E_{\pi_n}(\phi_0) - \frac{q-2}{n(q-1)} E_{\pi_n}(\phi_1) + \frac{n-1}{n} E_{\pi_n}(\phi_2) \\ &= \frac{1}{n(q-1)}. \end{aligned}$$

■

Next, we calculate the expectations of  $\phi_j$ 's and estimate the variance of  $\phi_1$  with respect to  $\nu_n^{*k}$  for any nonnegative integer  $k$ .

**Lemma 4.3** *Let  $n$  be a positive integer,  $q$  an integer with  $q \geq 2$  and  $k$  a nonnegative integer.*

(1) *For  $j \in \{0, 1, \dots, n\}$ , we have*

$$E_{\nu_n^{*k}}(\phi_j) = \left(1 - \frac{jq}{n(q-1)}\right)^k. \quad (4.3)$$

(2) *Assume that  $(n-2)(q-1) \geq 2$ . Then, one has*

$$\text{Var}_{\nu_n^{*k}}(\phi_1) \leq \frac{1}{n}. \quad (4.4)$$

*Proof.* (1) Since  $\phi_j$  is real-valued for any  $j \in \{0, 1, \dots, n\}$ , we have

$$\begin{aligned} E_{\nu_n^{*k}}(\phi_j) &= \sum_{x \in X_n} \phi_j(x) \nu_n^{*k}(x) = \sum_{x \in X_n} \nu_n^{*k}(x) \overline{\phi_j(x)} \\ &= \mathcal{F}(\nu_n^{*k})(\phi_j) = \mathcal{F}(\nu_n)(\phi_j)^k = \left(1 - \frac{jq}{n(q-1)}\right)^k. \end{aligned}$$

(2) By Lemma 4.1, we see that

$$\begin{aligned} \text{Var}_{\nu_n^{*k}}(\phi_1) &= E_{\nu_n^{*k}}(\phi_1^2) - E_{\nu_n^{*k}}(\phi_1)^2 \\ &= \frac{1}{n(q-1)} E_{\nu_n^{*k}}(\phi_0) + \frac{q-2}{n(q-1)} E_{\nu_n^{*k}}(\phi_1) + \frac{n-1}{n} E_{\nu_n^{*k}}(\phi_2) - E_{\nu_n^{*k}}(\phi_1)^2 \\ &= \frac{1}{n(q-1)} + \frac{q-2}{n(q-1)} \left(1 - \frac{q}{n(q-1)}\right)^k + \frac{n-1}{n} \left(1 - \frac{2q}{n(q-1)}\right)^k \\ &\quad - \left(1 - \frac{q}{n(q-1)}\right)^{2k}. \end{aligned}$$

Since  $n(q-1) - 2q = (n-2)(q-1) - 2 \geq 0$ , we see that

$$0 \leq 1 - \frac{2q}{n(q-1)} \leq \left(1 - \frac{q}{n(q-1)}\right)^2.$$

Hence, we have

$$\begin{aligned} \text{Var}_{\nu_n^{*k}}(\phi_1) &\leq \frac{1}{n(q-1)} + \frac{q-2}{n(q-1)} \left(1 - \frac{q}{n(q-1)}\right)^k - \frac{1}{n} \left(1 - \frac{q}{n(q-1)}\right)^{2k} \\ &\leq \frac{1}{n(q-1)} + \frac{q-2}{n(q-1)} \left(1 - \frac{q}{n(q-1)}\right)^k \leq \frac{1}{n(q-1)} + \frac{q-2}{n(q-1)} = \frac{1}{n}, \end{aligned}$$

since  $n(q-1) - q > n(q-1) - 2q \geq 0$ . ■

Now, we give a minorant function for total variance distance.

**Theorem 4.4** *Assume that  $q \geq 2$ . We fix a positive real number  $c_0 > 0$ . For any positive real number  $b > 0$ , there exists a positive integer  $n_0$  such that for any integer  $n$  with  $n \geq n_0$  and any integer  $k = \frac{n(q-1)}{2q}(\log n(q-1) - c)$  with  $0 \leq c \leq \min\{c_0, \log n(q-1)\}$ , we have*

$$\|\nu_n^{*k} - \pi_n\|_{TV} \geq 1 - (4q + b)e^{-c}. \quad (4.5)$$

*Proof.* We write  $n_1 = \max \left\{ \left\lceil \frac{e^{c_0}}{q-1} \right\rceil, 4 \right\}$  for simplicity, and assume that  $n \geq n_1 \geq 4$ . Then, we see that  $c_0 \leq \log n_1(q-1) \leq \log n(q-1)$ . Put

$$\beta_{n,k} = \sqrt{\frac{q}{(4q+b)(q-1)}} e^{\frac{c}{2}}. \quad (4.6)$$

Using  $\beta_{n,k}$ , we define a subset  $B_{n,k} \subset X_n$  of  $X_n$  by

$$B_{n,k} = \left\{ x \in X_n; |\phi_1(x)| < \frac{\beta_{n,k}}{\sqrt{n}} \right\}. \quad (4.7)$$

By Markov's inequality, we have

$$\begin{aligned} \pi_n(B_{n,k}) &= 1 - \pi_n \left( \left\{ x \in X_n; |\phi_1(x)| \geq \frac{\beta_{n,k}}{\sqrt{n}} \right\} \right) \\ &= 1 - \pi_n \left( \left\{ x \in X_n; \phi_1(x)^2 \geq \frac{\beta_{n,k}^2}{n} \right\} \right) \\ &\geq 1 - \frac{n}{\beta_{n,k}^2} E_{\pi_n}(\phi_1^2) = 1 - \frac{n}{\beta_{n,k}^2} \cdot \frac{1}{n(q-1)} = 1 - \frac{1}{\beta_{n,k}^2(q-1)}. \end{aligned}$$

We define a function  $\omega : [0, 1) \rightarrow \mathbb{R}$  on the interval  $[0, 1)$  by

$$\log(1-x) = -x - \frac{x^2}{2}\omega(x),$$

where  $x \in [0, 1)$ . Then,  $\omega(x) \geq 0$  for any  $x \in [0, 1)$ ,  $\lim_{x \rightarrow 0} \omega(x) = 1$  and we have

$$\begin{aligned} E_{\nu_n^{*k}}(\phi_1) &= \left( 1 - \frac{q}{n(q-1)} \right)^k = \exp \left( \log \left( 1 - \frac{q}{n(q-1)} \right)^k \right) \\ &= \exp \left( \log \left( 1 - \frac{q}{n(q-1)} \right) \cdot \frac{n(q-1)}{2q} (\log n(q-1) - c) \right) \\ &= \exp \left( \left( -\frac{q}{n(q-1)} - \frac{q^2}{2n^2(q-1)^2} \omega \left( \frac{q}{n(q-1)} \right) \right) \frac{n(q-1)}{2q} (\log n(q-1) - c) \right) \\ &= \frac{e^{\frac{c}{2}}}{\sqrt{n(q-1)}} \exp \left( \frac{q(c - \log n(q-1))}{4n(q-1)} \omega \left( \frac{q}{n(q-1)} \right) \right). \end{aligned}$$

Since  $0 \leq c \leq c_0 \leq \log n_1(q-1) \leq \log n(q-1)$  and  $\lim_{n \rightarrow \infty} \frac{\log n(q-1)}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , we see that

$$\begin{aligned} 0 &< \exp \left( -\frac{q \log n(q-1)}{4n(q-1)} \omega \left( \frac{q}{n(q-1)} \right) \right) \\ &\leq \exp \left( \frac{q(c - \log n(q-1))}{4n(q-1)} \omega \left( \frac{q}{n(q-1)} \right) \right) \\ &\leq \exp \left( \frac{q(c_0 - \log n(q-1))}{4n(q-1)} \omega \left( \frac{q}{n(q-1)} \right) \right) \leq 1, \\ \lim_{n \rightarrow \infty} \exp \left( \frac{-q \log n(q-1)}{4n(q-1)} \omega \left( \frac{q}{n(q-1)} \right) \right) &= 1. \end{aligned}$$

Hence, there exists an integer  $n_0$  with  $n_0 \geq n_1$  such that for any integer  $n$  with  $n \geq n_0$ , we have

$$\exp\left(\frac{-q \log n(q-1)}{4n(q-1)} \omega\left(\frac{q}{n(q-1)}\right)\right) \geq 2\sqrt{\frac{q}{4q+b}},$$

it implies that

$$E_{\nu_n^{*k}}(\phi_1) \geq 2\sqrt{\frac{q}{n(4q+b)(q-1)}} e^{\frac{c}{2}} = \frac{2\beta_{n,k}}{\sqrt{n}}.$$

So we have

$$B_{n,k} \subset B'_{n,k} = \left\{x \in X_n; |\phi_1(x) - E_{\nu_n^{*k}}(\phi_1)| \geq E_{\nu_n^{*k}}(\phi_1) - \frac{\beta_{n,k}}{\sqrt{n}}\right\}.$$

Hence, by Chebyshev's inequality,

$$\nu_n^{*k}(B_{n,k}) \leq \nu_n^{*k}(B'_{n,k}) \leq \frac{\text{Var}_{\nu_n^{*k}}(\phi_1)}{\left(E_{\nu_n^{*k}}(\phi_1) - \frac{\beta_{n,k}}{\sqrt{n}}\right)^2} \leq \frac{\frac{1}{n}}{\frac{\beta_{n,k}^2}{n}} = \frac{1}{\beta_{n,k}^2}.$$

Therefore

$$\begin{aligned} \|\nu_n^{*k} - \pi_n\|_{TV} &\geq \pi_n(B_{n,k}) - \nu_n^{*k}(B_{n,k}) \geq 1 - \frac{1}{\beta_{n,k}^2(q-1)} - \frac{1}{\beta_{n,k}^2} \\ &= 1 - \frac{q}{q-1} \cdot \frac{(4q+b)(q-1)}{q} e^{-c} = 1 - (4q+b)e^{-c}. \end{aligned}$$

■

The above theorem gives a simple lower bound of  $\|\nu_n^{*k} - \pi_n\|_{TV}$ .

**Corollary 4.5** Fix a positive real number  $c > 0$ . Put  $a_n = \frac{n(q-1)}{2q} \log n(q-1)$  and  $b_n = \frac{n(q-1)}{2q}$ . Then, we have

$$\liminf_{n \rightarrow \infty} \|\nu_n^{* \lfloor a_n - cb_n \rfloor} - \pi_n\|_{TV} \geq 1 - 4qe^{-c}. \quad (4.8)$$

**Remark 4.6** In Theorem 5.8 of [M2], Mizukawa gives a minorant function for random walks with staying on  $(K/L)^n$ , where  $(K, L)$  is a Gelfand pair. Our object is the case where  $(K, L) = (S_q, S_{q-1})$ ,  $a_0 = \frac{1}{q-1}$  and  $mp = 1$ . Put  $\gamma = 2\sqrt{\frac{q}{4q+b}}$ . Then we have that  $0 < \gamma < 1$  and that  $\delta = \frac{4(q-1)(a_0+1)}{\gamma^2} = 4q+b$ .

Katsuhiko Kikuchi  
Department of Mathematics  
Kyoto University  
606-8502 Kyoto, JAPAN  
e-mail: kikuchi@math.kyoto-u.ac.jp

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